Feynman path integral on the non-commutative plane

2003 J. Phys. A: Math. Gen. 36 L467

(http://iopscience.iop.org/0305-4470/36/33/101)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) L467–L471 PII: S0305-4470(03)64994-2

LETTER TO THE EDITOR

Feynman path integral on the non-commutative plane

Anais Smailagic[1] **and Euro Spallucci**[2]

[1] Sezione INFN di Trieste, Strada Costiera 11, 34014 Trieste, Italy
[2] Department of Theoretical Physics, University of Trieste, Strada Costiera 11, 34014 Trieste, Italy

E-mail: anais@ictp.trieste.it and spallucci@trieste.infn.it

Received 20 June 2003
Published 5 August 2003
Online at stacks.iop.org/JPhysA/36/L467

Abstract

We formulate the Feynman path integral on a non-commutative plane using coherent states. The propagator for a free particle exhibits UV cutoff induced by the parameter of non-commutativity.

PACS numbers: 11.10.Nx, 31.15.Kb

One of the difficulties when working with non-commutative geometries is to perform explicit calculations in terms of non-commutative coordinates x. In non-commutative quantum mechanics, it is possible to find a suitable combination of phase space coordinates which define new, *commuting* position coordinates, with non-commutativity manifesting itself as an external magnetic field [1, 2]. In quantum field theory, a similar approach cannot be applied since quantum fields are functions of position coordinates only, and momenta conjugate to these coordinates do not appear. In this case, it is customary to formulate a non-commutative field theory [3] in terms of ordinary functions of commuting variables, endowed with a ($*$-product) multiplication law [4]. However, in the formulation of non-commutative theory based on the $*$-product, the basic property of the non-commutativity, i.e. the existence of a natural UV, cutoff due to the uncertainty in position, is not transparent. In particular, the free propagator is unaffected by the $*$-product as if non-commutativity had no effect on it. Calculations are then performed using truncated series expansion in the parameter characterizing the non-commutativity of space. This approach leads to a UV/IR mixing phenomenon but one is still faced with divergences to cure [5] as in ordinary quantum field theory. It is possible that the complete resummation of the $*$-product expansion would give a finite result but such a procedure is beyond calculational capabilities (as it is in the ordinary perturbation approach). For this reason the use of the $*$-product, while a perfectly well-defined procedure, did not result in the expected UV finiteness of non-commutative field theory. On the other hand, the existence of a minimal length in the non-commutative plane should manifest itself already in the free propagator being the property of space (geometry) and not of interaction among fields.

With the hope of curing this flaw, we have recently presented a new formulation of quantum field theory on the non-commutative plane [6] which is based on coherent state formulation, instead of $*$-product, and which is explicitly UV finite.

In this letter, we would like to use the ideas introduced in [6] to formulate the Feynman path integral on the non-commutative plane. The reason is that the Feynman path integral provides a common framework both for quantum mechanics and field theory. As an example of this connection we shall calculate the Feynman propagator for a scalar particle as a sum over paths in phase space. Comparing the propagator found in this approach with that in [6] we find complete agreement.

The core ingredient in the formulation of non-commutative models without a $*$-product is the use of expectation values of operators between *coherent states*. In this way, non-commutativity of coordinates is carried on by the Gaussian spread of coherent states.

Contrary to space coordinates, we choose to consider *commutative* canonical momenta since their non-commutativity can be introduced through covariant derivatives in the usual way.

We start with a non-commutative two-dimensional plane described by the set of coordinates $\mathbf{q}_1$, $\mathbf{q}_2$ satisfying

$$[\mathbf{q}_1, \mathbf{q}_2] = \mathrm{i}\theta. \tag{1}$$

The main difficulty in working with non-commutative coordinates is that there are no common eigenstates of $\mathbf{q}_1$, $\mathbf{q}_2$ due to (1). Thus, one cannot work in coordinate representation of ordinary quantum mechanics. To bypass this problem we construct *new* raising/lowering operators in terms of coordinates only, as

$$\mathbf{A} \equiv \mathbf{q}_1 + \mathrm{i}\mathbf{q}_2 \tag{2}$$

$$\mathbf{A}^\dagger \equiv \mathbf{q}_1 - \mathrm{i}\mathbf{q}_2. \tag{3}$$

The coordinate analogue of the canonical commutator of creation and destruction operators is

$$[\mathbf{A}, \mathbf{A}^\dagger] = 2\theta. \tag{4}$$

The *Coherent states* [7] are defined as eigenstates of the above operators in the following sense:

$$\mathbf{A}|\alpha\rangle = \alpha|\alpha\rangle \tag{5}$$

$$\langle\alpha|\mathbf{A}^\dagger = \langle\alpha|\alpha^*. \tag{6}$$

The explicit form of the normalized coordinate coherent state is

$$|\alpha\rangle = \exp\left(-\tfrac{1}{2}\alpha\alpha^*\right)\exp(\alpha\mathbf{A}^\dagger)|0\rangle. \tag{7}$$

We define *expectation values* of non-commuting coordinates between coherent states as

$$\begin{aligned}\langle\alpha|\mathbf{q}_1|\alpha\rangle &= \langle\alpha|\frac{\mathbf{A}^\dagger + \mathbf{A}}{2}|\alpha\rangle \\ &= \frac{\alpha^* + \alpha}{2} = \mathrm{Re}\,(\alpha) \\ &\equiv x_1\end{aligned} \tag{8}$$

$$\begin{aligned}\langle\alpha|\mathbf{q}_2|\alpha\rangle &= \langle\alpha|\frac{-\mathbf{A}^\dagger + \mathbf{A}}{2\mathrm{i}}|\alpha\rangle \\ &= \frac{-\alpha^* + \alpha}{2\mathrm{i}} = \mathrm{Im}\,(\alpha) \\ &\equiv x_2.\end{aligned} \tag{9}$$

The vector $\vec{x} = (x_1, x_2)$ represents the *mean position* of the particle over the non-commutative plane. The advantage of the use of mean values is that they do not represent sharp eigenvalues and thus can be measured simultaneously, in spite of non-commutativity of coordinates. In the same spirit, with any operator valued function, $F(\mathbf{q}_1, \mathbf{q}_2)$, we associate a function $f(x_1, x_2)$.

For future purposes, let us consider the non-commutative version of the plane wave operator $\exp(\mathrm{i}(\vec{p} \cdot \vec{\mathbf{q}}))$ where $\vec{p} = (p_1 p_2)$ is a real two-component vector. The mean value is calculated to be

$$\langle \alpha | \mathrm{e}^{\mathrm{i}p_1 \mathbf{q}_1 + \mathrm{i}p_2 \mathbf{q}_2} | \alpha \rangle = \exp\left(-\theta \frac{\vec{p}^2}{4} + \mathrm{i}\vec{p} \cdot \vec{x} \right). \tag{10}$$

In order to obtain the above result we have exploited the Backer–Hausdorff decomposition

$$\begin{aligned} \mathrm{e}^{\mathrm{i}p_+ \mathbf{A}^\dagger + \mathrm{i}p_- \mathbf{A}} &= \mathrm{e}^{\mathrm{i}p_+ \mathbf{A}^\dagger}\, \mathrm{e}^{\mathrm{i}p_- \mathbf{A}}\, \mathrm{e}^{\frac{p_+ p_-}{2}[\mathbf{A}^\dagger, \mathbf{A}]} \\ &= \mathrm{e}^{\mathrm{i}p_+ \mathbf{A}^\dagger}\, \mathrm{e}^{\mathrm{i}p_- \mathbf{A}}\, \mathrm{e}^{-\theta p_+ p_-} \\ &= \mathrm{e}^{\mathrm{i}p_+ \mathbf{A}^\dagger}\, \mathrm{e}^{\mathrm{i}p_- \mathbf{A}}\, \mathrm{e}^{-\theta \frac{\vec{p}^2}{4}} \end{aligned} \tag{11}$$

where we found it convenient to define

$$p_- \equiv \frac{p_1 - \mathrm{i}p_2}{2} \tag{12}$$

$$p_+ \equiv \frac{p_1 + \mathrm{i}p_2}{2}. \tag{13}$$

The mean value of the plane wave, which takes into account the non-commutativity of coordinates, is the key to the Fourier transform of quantum fields in [6]. At the same time, (10) shows that the vector $\vec{p}$ is canonically conjugated to the *mean position* $\vec{x}$ and can be given the meaning of *mean linear momentum.* Thus, we can interpret (10) as the wavefunction of a 'free point particle' on the non-commutative plane:

$$\psi_{\vec{p}}(\vec{x}) \equiv \langle \vec{p} | \vec{x} \rangle_\theta \equiv \exp\left(-\theta \frac{\vec{p}^2}{4} + \mathrm{i}\vec{p} \cdot \vec{x} \right). \tag{14}$$

The amplitude between two states of different mean position is the key ingredient in the formulation of the Feynman path integral. It is given by

$$\begin{aligned} \langle \vec{y} | \vec{x} \rangle &= \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \langle \vec{y} | \vec{p} \rangle \langle \vec{p} | \vec{x} \rangle \\ &= \left(\frac{1}{2\pi\theta} \right) \exp\left(-\frac{(\vec{x} - \vec{y})^2}{2\theta} \right). \end{aligned} \tag{15}$$

We thus get the important result that the effect of non-commutativity in the scalar product between two mean positions is to substitute a Dirac δ-function by a Gaussian whose half-width $\sqrt{\theta}$ corresponds to the minimum length attainable in fuzzy space. Now, we can proceed and formulate the Feynman path integral for non-commutative theories.

One starts from the definition of discretized transition amplitude between two nearby points [8]. With the help of (14) we find

$$\begin{aligned} \langle \vec{x}_{(i+1)}, \epsilon | \vec{x}_{(i)}, 0 \rangle &= \langle \vec{x}_{(i+1)} | \mathrm{e}^{-\mathrm{i}\epsilon H} | \vec{x}_{(i)} \rangle \\ &= \langle \vec{x}_{(i+1)} | 1 - \mathrm{i}\epsilon H + O(\epsilon^2) | \vec{x}_{(i)} \rangle \\ &= \int \frac{\mathrm{d}^2 p_{(i)}}{(2\pi)^2} \langle \vec{x}_{(i+1)} \mid \vec{p}_{(i)} \rangle \langle \vec{p}_{(i)} \mid \vec{x}_{(i)} \rangle\, \mathrm{e}^{-\mathrm{i}\epsilon H(\vec{x}_{(i)}, \vec{p}_{(i)})} \\ &= \int \frac{\mathrm{d}^2 p_{(i)}}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}\left(\vec{x}_{(i+1)} - \vec{x}_{(i)}\right)\vec{p}_{(i)}} \exp\left(-\mathrm{i}\epsilon H(\vec{x}_{(i)}, \vec{p}_{(i)}) - \theta \vec{p}^{\,2}_{(i)}/2 \right). \end{aligned} \tag{16}$$

Following the usual steps of summation over discretized paths, and letting the number of intervals to infinity, one finds the non-commutative version of the path integral for the propagation kernel

$$K_\theta(x-y;T) = N\int[Dx][Dp]\exp\left\{\mathrm{i}\int_y^x \vec{p}\cdot\mathrm{d}\vec{x} - \int_0^T \mathrm{d}\tau\left(H(\vec{p},\vec{x}) + \frac{\theta}{2T}\vec{p}^2\right)\right\}. \tag{17}$$

The end result is that the path integral gets modified by the Gaussian factor produced by non-commutativity. To emphasize the physical relevance of this modification, let us calculate the propagator for the free particle on the non-commutative plane. Starting with (17) and performing the integration over coordinates, as in [9], we find

$$\begin{aligned} K_\theta(x-y;T) &= N\int[Dp]\delta[\dot{\vec{p}}]\exp\left\{\mathrm{i}[\vec{p}\cdot\vec{x}]_x^y - \int_0^T \mathrm{d}\tau\left(\frac{\vec{p}^2}{2m} + \frac{\theta}{2T}\vec{p}^2\right)\right\} \\ &= \int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\mathrm{e}^{\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})}\,\mathrm{e}^{-(T+m\theta)\vec{p}^2/2m}. \end{aligned} \tag{18}$$

The final form of the propagation kernel is

$$K_\theta(x-y;T) = \frac{1}{(2\pi)^2}\left(\frac{2\pi m}{T+m\theta}\right)\exp\left\{-\frac{m(\vec{x}-\vec{y})^2}{2(T+m\theta)}\right\}. \tag{19}$$

Let us verify the short-time limit of the propagation kernel. The result is

$$K_\theta(x-y;0) = \left(\frac{1}{2\pi\theta}\right)\exp\left\{-\frac{(\vec{x}-\vec{y})^2}{2\theta}\right\}. \tag{20}$$

It shows that the kernel is not a δ-function, but a Gaussian. This is due to the fact that the best possible localization of the particle, in non-commutative space, is within a cell of area θ.

Knowing the kernel, we can calculate the Green function. It is defined by the Laplace transform as

$$\begin{aligned} G_\theta(x-y;E) &\equiv \int_0^\infty \mathrm{d}T\,\mathrm{e}^{-ET}K_\theta(x-y;T) \\ &= \int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\mathrm{e}^{\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})}G_\theta(E;\vec{p}^2) \end{aligned} \tag{21}$$

where the momentum space Green function is given by

$$G_\theta(E;\vec{p}^2) = \left(\frac{1}{2\pi}\right)^2\frac{\exp(-\theta\vec{p}^2/2)}{E+\frac{\vec{p}^2}{2m}} \tag{22}$$

where $G_\theta(E;\vec{p}^2)$ shows an *exponential cutoff* for large momenta.

The above calculation refers to a non-relativistic free particle and a relativistic extension will be formulated in what follows. We are working in 2 + 1 dimensions and keep, as usual, the (Euclidean) time as a commuting variable. We would like to mention the fact that the non-commutative parameter θ selects preferred spatial directions leading to a violation of particle Lorentz invariance. This fact has already been addressed in non-commutative models [10] and we shall just follow the accepted wisdom. The relativistic version of $G_\theta(x-y;E)$ is

$$G(x-y;m^2) \equiv N\int[De][Dx][Dp]\exp\left\{\mathrm{i}\int_y^x p_\mu\,\mathrm{d}x^\mu - \int_0^T\mathrm{d}\tau\left[e(\tau)(p^2+m^2) + \frac{\theta}{2T}\vec{p}^2\right]\right\} \tag{23}$$

where $e(\tau)$ is a Lagrange multiplier enforcing mass shell condition for the relativistic particle:

$$G(x-y;m^2) = N\int[De]\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}p_\mu(x-y)^\mu}\exp\left\{-(p^2+m^2)\int_0^T\mathrm{d}\tau\,e(\tau) - \theta\vec{p}^2/2\right\}. \tag{24}$$

The integration over $x(\tau)$ and $p(\tau)$ is carried out as in the non-relativistic case. The integration of the Lagrange multiplier $e(\tau)$ is carried out by the introduction of the Feynman–Schwinger 'proper time' s,

$$1 = \int_0^\infty \mathrm{d}s\delta\left[s - \int_0^T \mathrm{d}\tau e(\tau)\right]. \tag{25}$$

The final result for the relativistic case is

$$\begin{aligned} G_\theta(x-y;m^2)s &= N\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-sm^2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}p_\mu(x-y)^\mu}\exp[-(s+\theta)\vec{p}^2/2] \\ &\equiv \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}p_\mu(x-y)^\mu}G_\theta(p^2;m^2) \end{aligned} \tag{26}$$

where the Feynman propagator in momentum space is given by

$$G_\theta(p^2;m^2) = \left(\frac{1}{2\pi}\right)^3 \frac{\exp[-\theta p^2/2]}{p^2+m^2}. \tag{27}$$

Thus, we arrive at the conclusion, already formulated in [6], that the non-commutativity of space leads to an exponential cutoff in the Green function at large momenta. The corresponding quantum field theory is UV finite since the loop diagrams exhibit no divergences due to the non-commutative cutoff.

References

[1] Nair V P and Polychronakos A P 2001 *Phys. Lett.* B **505** 267
[2] Smailagic A and Spallucci E 2001 *Phys. Rev.* D **65** 107701
Smailagic A and Spallucci E 2002 *J. Phys. A: Math. Gen.* **35** L363
[3] Alvarez-Gaume L and Wadia S R 2001 *Phys. Lett.* B **501** 319
Alvarez-Gaume L and Barbon J L F 2001 *Int. J. Mod. Phys.* A **16** 1123
[4] Weyl H 1927 *Z. Phys.* **46** 1
Wigner E P 1932 *Phys. Rev.* **40** 749
Moyal G E 1949 *Proc. Camb. Phil. Soc.* **45** 99
[5] Peet A W and Polchinski J 1999 *Phys. Rev.* D **59** 065011
[6] Smailagic A and Spallucci E UV divergence-free QFT on noncommutative plane *J. Phys. A: Math. Gen.* submitted
[7] Glauber R J 1963 *Phys. Rev.* **131** 2766
Klauder J R and Skagerstam B S (ed) 1985 *Coherent States: Applications to Physics and Mathematical Physics* (Singapore: World Scientific)
[8] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
[9] Ansoldi S, Aurilia A and Spallucci E 2000 *Eur. J. Phys.* **21** 1
[10] Carroll S M, Harvey J A, Kostelecky V A, Lane C D and Okamoto T 2001 *Phys. Rev. Lett.* **87** 141601